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Formelsammlung zur Vorlesung Kategoriale Daten

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Diese Formelsammlung darf in der Klausur verwendet werden. Eigene handschriftliche Notizen und Ergänzungen dürfen, ausschlielich auf den bedruckten Vorderseiten, eingefügt werden. Es drfen keine zustzlichen Bltter eingeheftet werden.

Contents

1 Distributions

1.1 Binomial distribution

The variable y is binomially distributed, $y \sim B(n,\pi)$, if the probability mass function is given by

$$f(y) = \begin{cases} \binom{n}{y}\pi^y(1-\pi)^{n-y} & y \in \{0,1,\dots,n\} \\ 0 & \text{otherwise}. \end{cases}$$

The parameters are $n \in \mathbb{N}, \pi \in [0,1]$. One obtains

$$E(y) = n\pi,\ \mathit{var}(y) = n\pi(1-\pi).$$

1.2 Multinomial distribution

The vector $\boldsymbol{y}^T = (y_1,\dots,y_k)$ is multinomially distributed, $\boldsymbol{y} \sim M(n,(\pi_1,\dots,\pi_k))$, if the probability mass function is given by

$$f(y_1,\dots,y_k) = \begin{cases} \frac{n!}{y_1!\cdot\ldots\cdot y_k!} \ \pi_1^{y_1}\cdot\ldots\cdot\pi_k^{y_k} & y_i \in \{0,\dots,n\},\ \sum_i y_i = n \\ 0 & \text{otherwise} \end{cases}$$

where $\boldsymbol{\pi}^T = (\pi_1,\dots,\pi_k)$ is a probability vector, i.e. $\pi_i \in [0,1],\ \sum_i \pi_i = 1$.
One has

$$E(y_i) = n\pi_i, \quad \mathit{var}(y_i) = n\pi_i(1-\pi_i),$$
$$\mathit{cov}(y_i,y_j) = -n\pi_i\pi_j, i \neq j.$$

1.3 Poisson distribution

A variable y follows a Poisson distribution, $y \sim Po(\lambda)$, if the probability mass function is given by

$$f(y) = \begin{cases} \frac{\lambda^y}{y!}e^{-\lambda} & y = 0,1,2,\dots \\ 0 & \text{otherwise}. \end{cases}$$

The parameter $\lambda > 0$ determines

$$E(y) = \lambda,\ \mathit{var}(y) = \lambda.$$

1.4 Poisson gamma distribution

The variable $y \in \{0, 1, \ldots\}$ follows a Poisson gamma distribution, $PoGa(\mu, a, b)$, (also known as negative binomial distribution), if the mass function is given by

$$f(y) = \frac{\Gamma(y+a)}{y!\,\Gamma(a)} \left(\frac{b}{b+\mu}\right)^a \left(\frac{\mu}{b+\mu}\right)^y,\ y \in \{0, 1, \ldots\}$$

where $\mu > 0, a > 0, b > 0$. One has

$$E(y) = \mu\frac{a}{b},\ \ \textit{var}(y) = \mu\frac{a}{b}\left(1 + \frac{\mu}{b}\right).$$

For $a \in \mathbb{N}$ the distribution is exclusively called negative binomial distribution, $NB(a, \pi)$, with mass function

$$f(y) = \binom{y+a-1}{y}\pi^a(1-\pi)^y,\ y \in \{0, 1, \ldots\}$$

and $\pi = b/(b+\mu)$. Here a denotes the number of successes and y the number of failures.

1.5 Gamma distribution

A random variable y is gamma distributed, $y \sim \Gamma(\nu, \alpha)$, if it has density function

$$f(y) = \begin{cases} 0 & y \le 0 \\ \frac{\alpha^\nu}{\Gamma(\nu)} & y^{\nu-1}e^{-\alpha y} \quad y > 0 \end{cases}$$

where for $\nu > 0$, $\Gamma(\nu)$ is defined by

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1}e^{-x}dx.$$

One has

$$E(y) = \frac{\nu}{\alpha},\ \ \textit{var}(y) = \frac{\nu}{\alpha^2}.$$

For $\nu = 1$ the exponential distribution is a special case.

2 Loglinear Models

2.1 Two-way tables

In general, two-way $(I \times J)$-contingency tables with I rows and J columns may be described by

$X_{ij} =$ counts in cell (i, j),
$X_A \in \{1, \ldots, I\}$ representing the rows,
$X_B \in \{1, \ldots, J\}$ representing the columns.

The observed contingency table has the form

$$\begin{array}{cc|cccc|c} & & & X_B & & & \\ & & 1 & 2 & \ldots & J & \\ \hline & 1 & X_{11} & X_{12} & \ldots & X_{1J} & X_{1+} \\ & 2 & X_{21} & \ddots & & \vdots & \vdots \\ X_A & \vdots & \vdots & & \ddots & \vdots & \\ & I & X_{I1} & \ldots & & X_{IJ} & X_{I+} \\ \hline & & X_{+1} & \ldots & & X_{+J} & \end{array}$$

where $X_{i+} = \sum_{j=1}^J X_{ij}, X_{+j} = \sum_{i=1} X_{ij}$ denote the marginal counts. The subscript "$+$" denotes the sum over that index.

Poisson and Multinomial settings

Let X_{ij}, $i = 1, \ldots, I$, $j = 1, \ldots, J$ follow independent Poisson distributions, $X_{ij} \sim Po(\lambda_{ij})$. Then the *conditional distribution* of $(X_{11}, \ldots, X_{IJ})$ given $n = \sum_{i,j} X_{ij}$ is multinomial. More concrete, one has

$$\left((X_{11}, \ldots, X_{IJ})|\sum_{i,j} X_{ij} = n\right) \sim M(n, (\lambda_{11}/\lambda, \ldots, \lambda_{IJ}/\lambda)),$$

with $\lambda = \sum_{i,j} \lambda_{ij}$.

Multinomial and Product-Multinomial settings

Let $(X_{11}, \ldots, X_{IJ})$ have multinomial distribution, $(X_{11}, \ldots, X_{IJ}) \sim M(n, (\pi_{11}, \ldots, \pi_{IJ}))$. By conditioning on the row margins $n_{i+} = \sum_j X_{ij}$ one obtains the product-multinomial distribution with probability mass function

$$f(x_{11}, \ldots, x_{IJ}) = \prod_{i=1}^{I} \frac{n_{i+}!}{x_{i1}! \cdot \ldots \cdot x_{iJ}!} \pi_{1|i}^{x_{i1}} \cdot \ldots \cdot \pi_{J|i}^{x_{iJ}}$$

where $\pi_{j|i} = \pi_{ij}/\sum_j \pi_{ij} = \pi_{ij}/\pi_{i+}$. Thus the cell counts of one row given n_{i+} have multinomial distribution,

$$(X_{i1}, \ldots, X_{iJ}) \sim M(n_{i+}, (\pi_{1|i}, \ldots, \pi_{J|i}))$$

Loglinear Model for Two-Way Tables

$$\log(\mu_{ij}) = \lambda_0 + \lambda_{A(i)} + \lambda_{B(j)} + \lambda_{AB(i,j)}$$

Constraints:

$$\sum_{i=1}^{I} \lambda_{A(i)} = \sum_{j=1}^{J} \lambda_{B(j)} = \sum_{i=1}^{I} \lambda_{AB(ij)} = \sum_{j=1}^{J} \lambda_{AB(ij)} = 0,$$

or

$$\lambda_{A(I)} = \lambda_{B(J)} = \lambda_{AB(iJ)} = \lambda_{AB(Ij)} = 0.$$

Additional constraints for multinomial tables:

$$\sum_{i,j} e^{\lambda_0} e^{\lambda_{A(i)}} e^{\lambda_{B(j)}} e^{\lambda_{AB(ij)}} = n$$

Additional constraints for product-multinomial tables:

$$\sum_{j=1}^{J} e^{\lambda_0} e^{\lambda_{A(i)}} e^{\lambda_{B(j)}} e^{\lambda_{AB(ij)}} = n_{i+}, \quad i = 1, \ldots, I \quad (\text{for } n_{i+} \text{ fixed}),$$

$$\sum_{i=1}^{I} e^{\lambda_0} e^{\lambda_{A(i)}} e^{\lambda_{B(j)}} e^{\lambda_{AB(ij)}} = n_{+j}, \quad j = 1, \ldots, J \quad (\text{for } n_{+j} \text{ fixed}).$$

2.2 Three-way tables

Three-Way Tables

X_A	X_B	X_C: 1	2	$\cdots$	K	
1	1	X_{111}	X_{112}	$\cdots$	X_{11K}	X_{11+}
	2	X_{121}	X_{122}			$\vdots$
	$\vdots$	$\vdots$				
	J	X_{1J1}	$\cdots$		X_{1JK}	X_{1J+}
2	1	X_{211}	X_{212}	$\cdots$	X_{21K}	X_{21+}
	2	X_{221}	X_{222}			$\vdots$
	$\vdots$	$\vdots$				
	J	X_{2J1}	$\cdots$		X_{2JK}	X_{2J+}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
I	1	X_{I11}	X_{I12}	$\cdots$	X_{I1K}	X_{I1+}
	2	X_{I21}	X_{I22}			$\vdots$
	$\vdots$	$\vdots$				
	J	X_{IJ1}	$\cdots$		X_{IJK}	X_{IJ+}

Product-Multinomial settings

$n_{i++} = X_{i++}$ is fixed and

$$(X_{i11}, \ldots, X_{iJK}) \sim M(n_{i++}, (\pi_{i11}, \ldots, \pi_{iJK})), \qquad (1)$$

where $\pi_{ijk} = P(X_B = j, X_C = k | X_A = i)$. An example of the second variant (two design variables) is obtained by letting X_A and X_B be design variables, i. e. $n_{ij+} = X_{ij+}$ is fixed and

$$(X_{ij1}, \ldots, X_{ijK}) \sim M(n_{ij+}, (\pi_{ij1}, \ldots, \pi_{ijK})). \qquad (2)$$

where $\pi_{ijk} = P(X_C = k | X_A = i, X_B = j)$.

Loglinear Model for Three-Way Tables

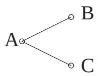
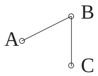
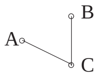
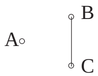
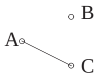
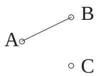
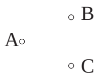
$$\log(\mu_{ijk}) = \lambda_0 + \lambda_{A(i)} + \lambda_{B(j)} + \lambda_{C(k)} + \lambda_{AB(i,j)} + \lambda_{AC(ik)} + \lambda_{BC(jk)} + \lambda_{ABC(ijk)}$$

Constraints:

$$\sum_i \lambda_{A(i)} = \sum_j \lambda_{B(j)} = \sum_k \lambda_{C(k)} = 0,$$
$$\sum_i \lambda_{AB(ij)} = \sum_j \lambda_{AB(ij)} = \sum_i \lambda_{AC(ik)} = \sum_k \lambda_{AC(ik)} = \sum_j \lambda_{BC(jk)} = \sum_k \lambda_{BC(jk)} = 0,$$
$$\sum_i \lambda_{ABC(ijk)} = \sum_j \lambda_{ABC(ijk)} = \sum_k \lambda_{ABC(ijk)} = 0,$$

or

$$\lambda_{A(I)} = \lambda_{B(J)} = \lambda_{C(K)} = 0,$$
$$\lambda_{AB(i,J)} = \lambda_{AB(I,j)} = \lambda_{AC(iK)} = \lambda_{AC(Ik)} = \lambda_{BC(jK)} = \lambda_{BC(Jk)} = 0,$$
$$\lambda_{ABC(Ijk)} = \lambda_{ABC(iJk)} = \lambda_{ABC(ijK)} = 0.$$

	Loglinear Model	Regressors of Logit-Model
AB/AC X_B, X_C conditionally independent, given X_A		$1, x_A$
AB/BC X_A, X_C conditionally independent, given X_B		$1, x_B$
AC/BC X_A, X_B conditionally independent, given X_C		$1, x_A, x_B$
A/BC X_A independent of (X_B, X_C)		$1, x_B$
AC/B (X_A, X_C) independent of X_B		$1, x_A$
AB/C (X_A, X_B) independent of X_C		1
$A/B/C$ X_A, X_B, X_C are dependent		1

AB/AC	$\lambda_{ABC} = \lambda_{BC} = 0$	$P(X_B, X_C \mid X_A) = P(X_B \mid X_A)P(X_C \mid X_A)$ X_B, X_C conditionally independent given X_A
AB/BC	$\lambda_{ABC} = \lambda_{AC} = 0$	$P(X_A, X_C \mid X_B) = P(X_A \mid X_B)P(X_C \mid X_B)$ X_A, X_C conditionally independent given X_B
AC/BC	$\lambda_{ABC} = \lambda_{AB} = 0$	$P(X_A, X_B \mid X_C) = P(X_A \mid X_C)P(X_B \mid X_C)$ X_A, X_B conditionally independent given X_C
A/BC	$\lambda_{ABC} = \lambda_{AB} = \lambda_{AC} = 0$	$P(X_A, X_B, X_C) = P(X_A)P(X_B, X_C)$ X_A jointly independent of (X_B, X_C)
AC/B	$\lambda_{ABC} = \lambda_{AB} = \lambda_{BC} = 0$	$P(X_A, X_B, X_C) = P(X_A, X_C)P(X_B)$ (X_A, X_C) jointly independent of X_B
AB/C	$\lambda_{ABC} = \lambda_{AC} = \lambda_{BC} = 0$	$P(X_A, X_B, X_C) = P(X_A, X_B)P(X_C)$ (X_A, X_B) jointly independent of X_C
$A/B/C$	$\lambda_{ABC} = \lambda_{AB} = \lambda_{AC} =$ $\lambda_{BC} = 0$	$P(X_A, X_B, X_C) = P(X_A)P(X_B)P(X_C)$ X_A, X_B, X_C are independent

Type 0: Saturated Model

$$\log(\mu_{ijk}) = \lambda_0 + \lambda_{A(i)} + \lambda_{B(j)} + \lambda_{C(k)} + \lambda_{AB(i,j)} + \lambda_{AC(ik)} + \lambda_{BC(jk)} + \lambda_{ABC(ijk)}.$$

Type 1: No three-factor interaction

$$\log(\mu_{ijk}) = \lambda_0 + \lambda_{A(i)} + \lambda_{B(j)} + \lambda_{C(k)} + \lambda_{AB(ij)} + \lambda_{AC(ik)} + \lambda_{BC(jk)}.$$

Type 2: Only two two-factor interactions contained

$$\log(\mu_{ijk}) = \lambda_0 + \lambda_{A(i)} + \lambda_{B(j)} + \lambda_{C(k)} + \lambda_{AC(ik)} + \lambda_{BC(jk)}.$$

Type 3: Only one two-factor interaction contained

$$\log(\mu_{ijk}) = \lambda_0 + \lambda_{A(i)} + \lambda_{B(j)} + \lambda_{C(k)} + \lambda_{BC(jk)}.$$

Type 4: Main effects model

$$\log(\mu_{ijk}) = \lambda_0 + \lambda_{A(i)} + \lambda_{B(j)} + \lambda_{C(k)}.$$

2.3 Inference for Loglinear Models

Likelihood and log-likelihood

Let all the Parameters be collected in one parameter vector γ. From the likelihood function

$$L(\boldsymbol{\gamma}) = \prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{k=1}^{K} \frac{\mu_{ijk}^{x_{ijk}}}{x_{ijk}!} e^{-\mu_{ijk}}$$

one obtains the log-likelihood

$$l(\boldsymbol{\gamma}) = \sum_{i,j,k} x_{ijk} \log(\mu_{ijk}) - \sum_{i,j,k} \mu_{ijk} - \sum_{i,j,k} \log(x_{ijk}!).$$

With μ_{ijk} parameterized as the saturated loglinear model one obtains by rearranging terms (and omitting constants)

$$\begin{aligned} l(\gamma) &= n\lambda_0 + \sum_i x_{i++}\lambda_{A(i)} + \sum_j x_{+j+}\lambda_{B(j)} + \sum_k x_{++k}\lambda_{C(k)} \\ &+ \sum_{i,j} x_{ij+}\lambda_{AB(ij)} + \sum_{i,k} x_{i+k}\lambda_{AC(ik)} + \sum_{j+k} x_{+jk}\lambda_{BC(jk)} \\ &+ \sum_{i,j,k} x_{ijk}\lambda_{AB(ijk)} - \sum_{i,j,k} \exp(\lambda + \lambda_{A(i)} + \ldots + \lambda_{ABC(ijk)}). \end{aligned}$$

Testing and Goodness of fit

For models with an intercept the deviance has the form

$$D = 2\sum_i x_i \log\left(\frac{x_i}{\hat{\mu}_i}\right).$$

When considering goodness-of-fit an alternative is Pearson's χ^2

$$\chi^2_P = \sum_i \frac{(x_i - \hat{\mu}_i)^2}{\hat{\mu}_i}.$$

For fixed N, both statistics have approximate χ^2-distribution if the assumed model holds and means μ_i are large. The degrees of freedom are $N - p$ where p is the number of estimated parameters.

3 Regression Models

3.1 Binary Response

$$\pi(\boldsymbol{x}_i) = P(y_i = 1|\boldsymbol{x}_i) = h(\boldsymbol{x}_i^T\boldsymbol{\beta})$$

Likelihood and Fisher Matrix for grouped data (with N groups)

$$l(\boldsymbol{\beta}) = \sum_{i=1}^N n_i \left\{\bar{y}_i \log\left(\frac{\pi(\boldsymbol{x}_i)}{1-\pi(\boldsymbol{x}_i)}\right) + \log(1-\pi(\boldsymbol{x}_i))\right\}$$

$$F(\boldsymbol{\beta}) = E(-\partial l^2(\boldsymbol{\beta})/\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}^T) = \sum_{i=1}^N n_i \frac{\partial h(\boldsymbol{x}_i^T\boldsymbol{\beta})/\partial\eta}{h(\boldsymbol{x}_i^T\boldsymbol{\beta})(1-h(\boldsymbol{x}_i^T\boldsymbol{\beta}))}\boldsymbol{x}_i\boldsymbol{x}_i^T.$$

For the logit model which has canonical link one obtains the simpler Fisher matrix

$$F(\boldsymbol{\beta}) = \sum_{i=1}^N n_i h(\boldsymbol{x}_i^T\boldsymbol{\beta})(1-h(\boldsymbol{x}_i^T\boldsymbol{\beta}))\boldsymbol{x}_i\boldsymbol{x}_i^T.$$

Alternative links:

Probit model

$$\pi(\boldsymbol{x}) = \phi(\boldsymbol{x}^T\boldsymbol{\beta});\ \ \phi^{-1}(\pi(\boldsymbol{x})) = \boldsymbol{x}^T\boldsymbol{\beta}.$$

Complementary log-log model

$$\pi(\boldsymbol{x}) = 1 - \exp(-\exp(\boldsymbol{x}^T\boldsymbol{\beta}));\ \ \log(-\log(1-\pi(\boldsymbol{x}))) = \boldsymbol{x}^T\boldsymbol{\beta}.$$

Log-log model

$$\pi(\boldsymbol{x}) = \exp(-\exp(-\boldsymbol{x}^T\boldsymbol{\beta}));\ \ -\log(-\log(\pi(\boldsymbol{x}))) = \boldsymbol{x}^T\boldsymbol{\beta}.$$

Exponential distribution or Complementary log model

$$\pi(\boldsymbol{x}) = 1 - \exp(-\boldsymbol{x}^T\boldsymbol{\beta});\ \ -\log(1-\pi(\boldsymbol{x})) = \boldsymbol{x}^T\boldsymbol{\beta}.$$

Exponential or log-link model

$$\pi(\boldsymbol{x}) = \exp(\boldsymbol{x}^T\boldsymbol{\beta});\ \ \log(\pi(\boldsymbol{x})) = \boldsymbol{x}^T\boldsymbol{\beta}.$$

Cauchy model

$$\pi(\boldsymbol{x}) = \tan^{-1}(\boldsymbol{x}^T\boldsymbol{\beta})/\pi + 1/2;\ \ \tan(\pi(\pi(\boldsymbol{x}) - 1/2)) = \boldsymbol{x}^T\boldsymbol{\beta}.$$

Explanatory value

Likelihood-Ratio-Index

$$R^2_{MF} = 1 - \frac{\log(L(\hat{\boldsymbol{\beta}}))}{\log(L(\hat{\beta}_0))}$$

Cox & Snell (1989)

$$R^2_{LR} = 1 - \left(\frac{L(\hat{\beta}_0)}{L(\hat{\boldsymbol{\beta}})}\right)^{2/n}$$

Efrons measure

$$R^2_E = 1 - \frac{\sum_{i=1}^n (y_i - \hat{\pi}_i)^2}{\sum_{i=1}^n (y_i - \bar{y})^2}$$

Mittlere Vorhersagedifferenz

$$\lambda = \frac{1}{n_1}\sum_{y_i=1}\hat{\pi}_i - \frac{1}{n_2}\sum_{y_i=0}\hat{\pi}_i$$

Kendalls

$$\tau_a = \frac{N_c - N_d}{n(n-1)/2}$$

Overdispersion: Beta-binomial Model

$$P(y_i; n_i, a_i, b_i) = \int P(y_i | D_i = \vartheta_i) p(\vartheta_i) d\vartheta_i$$
$$= \int \binom{n_i}{y_i} \vartheta_i^{y_i} (1-\vartheta_i)^{n_i - y_i} \frac{\Gamma(a_i + b_i)}{\Gamma(a_i)\Gamma(b_i)} \vartheta_i^{a_i - 1} (1-\vartheta_i)^{b_i - 1} d\vartheta_i$$
$$= \binom{n_i}{y_i} \frac{(a_i + y_i - 1)_{y_i} (b_i + n_i - y_i - 1)_{n_i - y_i}}{(a_i + b_i + n_i - 1)_{n_i}}.$$

Overdispersion: Quasi-likelihood

$$E(y_i) = n_i \pi_i = n_i h(\mathbf{x}_i^T \beta),$$
$$var(y_i) = n_i \pi_i (1 - \pi_i) \phi.$$

3.2 Multinomial Response

Generic Multinomial Logit Model

$$P(Y = r | \boldsymbol{x}) = \frac{\exp(\boldsymbol{x}^T \boldsymbol{\beta}_r)}{\sum_{s=1}^k \exp(\boldsymbol{x}^T \boldsymbol{\beta}_s)}$$

with optional side constraints

$$\boldsymbol{\beta}_k = (0, \ldots, 0)^T \quad \text{reference category } k$$
$$\boldsymbol{\beta}_{r_0} = (0, \ldots, 0)^T \quad \text{reference category } r_0$$
$$\sum_{s=1}^k \boldsymbol{\beta}_s = (0, \ldots, 0)^T \quad \text{symmetric side constraint}$$

Multinomial logit model mit reference category *k*

$$\log \left(\frac{P(Y = r|\boldsymbol{x})}{P(Y = k|\boldsymbol{x})} \right) = \boldsymbol{x}^T \boldsymbol{\beta}_r, \quad r = 1, \ldots, k-1,$$

or

$$P(Y = r|\boldsymbol{x}) = \frac{\exp(\boldsymbol{x}^T \boldsymbol{\beta}_r)}{1 + \sum_{s=1}^{k-1} \exp(\boldsymbol{x}^T \boldsymbol{\beta}_s)}, \quad r = 1, \ldots, k-1,$$
$$P(Y = k|\boldsymbol{x}) = \frac{1}{1 + \sum_{s=1}^{k-1} \exp(\boldsymbol{x}^T \boldsymbol{\beta}_s)}.$$

Multinomial logit model with category specific covariates (reference category *k*)

$$P(Y = r|\boldsymbol{x}, \{\boldsymbol{w}_j\}) = \frac{\exp(\boldsymbol{x}^T \boldsymbol{\beta}_r + \boldsymbol{v}_r^T \boldsymbol{\alpha})}{1 + \sum_{s=1}^{k-1} \exp(\boldsymbol{x}^T \boldsymbol{\beta}_s + \boldsymbol{v}_s^T \boldsymbol{\alpha})}, \quad r = 1, \ldots, k-1$$
$$P(Y = k|\boldsymbol{x}, \{\boldsymbol{w}_j\}) = \frac{1}{1 + \sum_{s=1}^{k-1} \exp(\boldsymbol{x}^T \boldsymbol{\beta}_s + \boldsymbol{v}_s^T \boldsymbol{\alpha})}$$

or

$$\log \left(\frac{P(Y = r|\boldsymbol{x}, \{\boldsymbol{w}_i\})}{P(Y = k|\boldsymbol{x}, \{\boldsymbol{w}_i\})} \right) = \boldsymbol{x}^T \boldsymbol{\beta}_r + \boldsymbol{v}_r^T \boldsymbol{\alpha}, \quad r = 1, \ldots, k-1$$

3.3 Multinomial Response: Ordered Categories

Cumulative Type Model, dichotomization into groups

$$[1, \ldots, r | r+1, \ldots, k] \qquad y_r = \begin{cases} 1 & Y \in \{1, \ldots, r\} \\ 0 & Y \in \{r+1, \ldots, k\} \end{cases}$$

Sequential Type Model, dichotomization given $Y \geq r$

$$1, \ldots, [r | r+1, \ldots, k] \qquad y_r = \begin{cases} 1 & Y = r \\ 0 & Y > r \end{cases} \quad \text{given } Y \geq r$$

Adjacent Type Model, dichotomization given $Y \in \{r, r+1\}$

$$1, \ldots, [r | r+1], \ldots, k] \qquad y_r = \begin{cases} 1 & Y = r \\ 0 & Y \geq r+1 \end{cases} \quad \text{given } Y \in \{r, r+1\}$$

Threshold (simple Cumulative) Model

$$P(Y \leq r|\boldsymbol{x}) = F(\gamma_{0r} + \boldsymbol{x}^T \boldsymbol{\gamma}),$$

or

$$P(Y = r|\boldsymbol{x}) = F(\gamma_{0r} + \boldsymbol{x}^T \boldsymbol{\gamma}) - F(\gamma_{0,r-1} + \boldsymbol{x}^T \boldsymbol{\gamma}),$$

where $-\infty = \gamma_{00} \leq \gamma_{01} \leq \cdots \leq \gamma_{0k} = \infty$.

Sequential Model

$$P(Y = r | Y \geq r, \boldsymbol{x}) = F(\gamma_{0r} + \boldsymbol{x}^T \boldsymbol{\gamma}), \quad r = 1, \ldots, k,$$

or

$$P(Y = r|\boldsymbol{x}) = F(\gamma_{0r} + \boldsymbol{x}^T \boldsymbol{\gamma}) \prod_{i=1}^{r-1} (1 - F(\gamma_{0i} + \boldsymbol{x}^T \boldsymbol{\gamma})).$$

Cumulative Logit Model (Proportional Odds Model)

$$P(Y \leq r|\boldsymbol{x}) = \frac{\exp(\gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma})}{1 + \exp(\gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma})},$$

or

$$\log\left(\frac{P(Y \leq r|\boldsymbol{x})}{P(Y > r|\boldsymbol{x})}\right) = \gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma}.$$

Cumulative Probit Model

$$P(Y \leq r|\boldsymbol{x}) = \Phi(\gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma}).$$

Cumulative Maximum Extreme Value (Gumbel) Model

$$P(Y \leq r|\boldsymbol{x}) = \exp(-\exp(\gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma})),$$

or

$$\log(-\log(P(Y \leq r|\boldsymbol{x})) = \gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma}.$$

Cumulative Minimum Extreme Value Model (Proportional Hazards Model, Gompertz Model)

$$P(Y \leq r|\boldsymbol{x}) = 1 - \exp(-\exp(\gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma})),$$

or

$$\log(-\log(P(Y > r|\boldsymbol{x})) = \gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma},$$

or

$$P(Y = r|Y \geq r, \boldsymbol{x}) = 1 - \exp(-\exp(\tilde{\gamma}_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma})),$$

with $\gamma_{0r} = \log(\sum_{i=1}^{r}\exp(\tilde{\gamma}_{0i}))$, $r = 1, \ldots, k$.

Sequential Logit Model (Continuation Ratio Logits Model)

$$P(Y = r|Y \geq r, \boldsymbol{x}) = \frac{\exp(\gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma}_r)}{1 + \exp(\gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma})},$$

or

$$\log\left(\frac{P(Y = r|\boldsymbol{x})}{P(Y > r|\boldsymbol{x})}\right) = \gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma}.$$

Generalized Cumulative Model

$$P(Y \leq r|\boldsymbol{x}) = F(\gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma}_r), \quad r = 1, \ldots, k.$$

Generalized Sequential Model

$$P(Y = r|Y \geq r, \boldsymbol{x}) = F(\gamma_{0r} + \boldsymbol{x}^T\boldsymbol{\gamma}_r), \quad r = 1 \ldots, k.$$

3.4 Common structure

Basic structure of multicategorical models

$$g(\boldsymbol{\pi}_i) = \mathbf{Z}_i\boldsymbol{\beta}, \quad \boldsymbol{\pi}_i = h(\mathbf{Z}_i\boldsymbol{\beta}),$$

with

$$\begin{aligned} \boldsymbol{g} &= (g_1, \ldots, g_q)^T : \mathbb{R}^q \longrightarrow \mathbb{R}^q, \quad h = g^{-1}, \\ \boldsymbol{\pi}_i &= (\pi_{i1}, \ldots, \pi_{iq})^T, \quad \pi_{ir} = P(Y = r|\mathbf{x}_i). \end{aligned}$$

3.5 Inference

Log Likelihood

$$n_i\mathbf{p}_i \sim M(n_i, \boldsymbol{\pi}_i), i = 1, \ldots, g$$

$$l(\boldsymbol{\beta}) = \sum_{i=1}^{g}\left\{\sum_{r=1}^{q} n_i p_{ir}\log(\pi_{ir}/(1 - \pi_{i1} - \cdots - \pi_{iq})) + \log(1 - \pi_{i1} - \cdots - \pi_{iq})\right\} + \log(c(p_{i1} \ldots, p_{iq}))$$

Score function

$$s(\boldsymbol{\beta}) = \partial l(\boldsymbol{\beta})/\partial\boldsymbol{\beta} = \sum_{i=1}^{g}\mathbf{Z}_i^T\mathbf{D}_i(\boldsymbol{\beta})\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{\beta})(\mathbf{p}_i - \boldsymbol{\mu}_i(\boldsymbol{\beta})),$$

where $\mathbf{D}_i(\boldsymbol{\beta}) = \partial h(\mathbf{Z}_i\boldsymbol{\beta})/\partial\boldsymbol{\eta}, \boldsymbol{\Sigma}_i(\boldsymbol{\beta}) = cov(\mathbf{p}_i) = \frac{1}{n_i}\left\{\mathrm{diag}(\boldsymbol{\pi}_i) - \boldsymbol{\pi}_i\boldsymbol{\pi}_i^T\right\}$

Fisher matrix

$$F(\boldsymbol{\beta}) = \sum_{i=1}^{g}\mathbf{Z}_i^T\mathbf{W}_i(\boldsymbol{\beta})\mathbf{Z}_i,$$

where $\mathbf{W}_i(\beta) = \mathbf{D}_i(\boldsymbol{\beta})\boldsymbol{\Sigma}_i(\boldsymbol{\beta})^{-1}\mathbf{D}_i(\boldsymbol{\beta})$

Goodness-of-fit Tests

Pearson Statistic

$$\chi^2_P = \sum_{i=1}^{g} \chi^2_P(\boldsymbol{p}_i, \hat{\boldsymbol{\pi}}_i),$$

$$\text{with} \quad \chi^2_P(\boldsymbol{p}_i, \hat{\boldsymbol{\pi}}_i) = n_i \sum_{r=1}^{k} (p_{ir} - \hat{\pi}_{ir})^2 / \hat{\pi}_{ir}.$$

Deviance

$$\chi^2_D = \sum_{i=1}^{g} \chi^2_D(\boldsymbol{p}_i, \hat{\boldsymbol{\pi}}_i),$$

$$\text{with} \quad \chi^2_D(\boldsymbol{p}_i, \hat{\boldsymbol{\pi}}_i) = 2n_i \sum_{r=1}^{k} \pi_{ir} \log\left(\frac{p_{ir}}{\hat{\pi}_{ir}}\right).$$

4 Regression Analysis of count data

4.1 Basic Model

$$P(Y = y) = \begin{cases} \frac{\lambda^y}{y!} e^{-\lambda} & \text{for } y \in \{0, 1, 2, \dots\} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_i = \exp(\boldsymbol{x}_i^T \boldsymbol{\beta}), \quad \log(\mu_i) = \boldsymbol{x}_i^T \boldsymbol{\beta}$$

Log-likelihood $l(\boldsymbol{\beta})$, score function $s(\boldsymbol{\beta})$ and Fisher matrix $F(\boldsymbol{\beta})$:

$$l(\boldsymbol{\beta}) = \sum_{i=1}^{n} y_i \boldsymbol{x}_i^T \boldsymbol{\beta} - \exp(\boldsymbol{x}_i^T \boldsymbol{\beta}) + \log(y_i!),$$

$$s(\boldsymbol{\beta}) = \sum_{i=1}^{n} \boldsymbol{x}_i (y_i - \exp(\boldsymbol{x}_i^T \boldsymbol{\beta})),$$

$$F(\boldsymbol{\beta}) = \sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i^T \exp(\boldsymbol{x}_i^T \boldsymbol{\beta}).$$

Goodness-of-fit for Poisson Regression Model

$$D = 2 \sum_{i=1}^{N} y_i \log\left(\frac{y_i}{\hat{\mu}_i}\right),$$

$$\chi^2_P = \sum_{i=1}^{N} \frac{(y_i - \hat{\mu}_i)^2}{\hat{\mu}_i}.$$

For $\mu_i \to \infty$ one obtains the approximation

$$D, \chi^2_P \sim \chi^2(N - p).$$

4.2 Modelling of Overdispersion

(1) Quasi-Likelihood Methods

$$\mu_i = h(x_i^T \beta)$$

Estimation equation

$$\sum_{i=1}^{n} \boldsymbol{x}_i \frac{\partial \mu_i}{\partial \eta} \frac{y_i - \mu_i}{v(\mu_i)} = 0$$

(2) Negative binomial Gamma-Poisson Model

$$\begin{aligned} P(y_i) &= \int f(y_i|b_i) f(b_i) \boldsymbol{d}\, b_i \\ &= \int \left(e^{-b_i \mu_i} \frac{(b_i \mu_i)^{y_i}}{y_i!} \right) \left(\frac{\nu^\nu}{\Gamma(\nu)} b_i^{\nu-1} e^{-\nu b_i} \right) \boldsymbol{d}\, b_i \\ &= \frac{\Gamma(y_i + \nu)}{\Gamma(\nu)\Gamma(y_i + 1)} \left(\frac{\mu_i}{\mu_i + \nu} \right)^{y_i} \left(\frac{\nu}{\mu_i + \nu} \right)^{\nu}. \end{aligned}$$

$$E(y_i) = \mu_i = \exp(\boldsymbol{x}_i^T \boldsymbol{\beta}), \quad var(y_i) = \mu_i + \frac{1}{\nu} \mu_i^2.$$

(3) Zero-inflated counts

With C denoting the class indicator of subpopulations ($C_i = 1$ for responders and $C_i = 0$ for non-responders) one obtains the mixture distribution

$$P(y_i = y) = P(y_i = y | C_i = 1)\pi_i + P(y_i = y | C_i = 0)(1 - \pi_i)$$

$$\begin{aligned} E(y_i) &= \pi_i \mu_i, \\ var(y_i) &= \pi_i \mu_i + \pi_i (1 - \pi_i) \mu_i^2 \\ &= \pi_i \mu_i (1 + \mu_i (1 - \pi_i)). \end{aligned}$$

$$\begin{aligned} \log(\mu_i) &= \boldsymbol{x}_i^T \boldsymbol{\beta}, \\ \text{logit}(\pi_i) &= \boldsymbol{z}_i^T \boldsymbol{\gamma}, \end{aligned}$$

(4) Hurdle Models

$$\begin{aligned} P(y = 0) &= f_1(0), \\ P(y = r) &= f_2(r) \frac{1 - f_1(0)}{1 - f_2(0)}, \quad r = 1, 2, \ldots \end{aligned}$$

$$\begin{aligned} E(y) &= \gamma \mu_2, \\ var(y) &= \sum_{r=1}^{n} r^2 f_2(r) \gamma - \left(\sum_{r=1}^{\infty} r\, f_2(r) \gamma \right)^2 \\ &= \mu_2 (1 + \mu_2) \gamma - \mu_2^2 \gamma^2. \end{aligned}$$

5 Repeated Measurements

5.1 Tests for binary response

General form

		y_2 1	0
y_1	1	n_{11}	n_{10}
	0	n_{01}	n_{00}

Hypotheses

$$H_0 : P(y_2 = 1) - P(y_1 = 1) = 0$$
$$H_1 : P(y_2 = 1) - P(y_1 = 1) \neq 0$$

(1) McNemar

$$M = \frac{(n_{01} - n_{10})^2}{n_{01} + n_{10}} \overset{H_0}{\underset{a}{\sim}} \chi^2(1)$$

(2) Binomial test

$$n_{01} | n_{01} + n_{10} = t \overset{H_0}{\sim} B(t, 0.5)$$

(3) Likelihood Ratio test

$$\lambda = 2n_{01} \log\left(\frac{2n_{01}}{n_{01} + n_{10}}\right) + 2n_{10} \log\left(\frac{2n_{10}}{n_{01} + n_{10}}\right) \overset{H_0}{\underset{a}{\sim}} \chi^2(1)$$

5.2 Logistic Regression

(a) **Subject-specific parameter**

$$\text{logit}\left(P(y_{it} = 1|x_t)\right) = \beta_i + x_t\beta, \quad x_t = \begin{cases} 1, & t = 2 \\ 0, & t = 1 \end{cases}$$

Conditional likelihood

$$L_c(\beta) = \prod_{y_{i1}+y_{i2}=1} P\left(y_{i1}, y_{i2} | y_{i1} + y_{i2} = 1\right) = \left(\frac{e^\beta}{1 + e^\beta}\right)^{n_{01}} \left(\frac{1}{1 + e^\beta}\right)^{n_{10}}$$

Hypotheses

$$H_0 : \beta = 0 \qquad H_1 : \beta \neq 0$$

Test statistic

$$\frac{\hat{\beta}}{\sqrt{\frac{1}{n_{01}} + \frac{1}{n_{10}}}} \overset{a}{\sim} N(0, 1)$$

(b) **Regression with covariates**

$$\text{logit}\left(P(y_{it} = 1|\mathbf{x}_{it})\right) = \beta_i + \mathbf{x}_{it}^T\boldsymbol{\gamma}$$

Conditional likelihood

$$\begin{aligned} L_c(\boldsymbol{\gamma}) &= \prod_{y_{i1}+y_{i2}=1} P\left(y_{i1}, y_{i2} | y_{i1} + y_{i2} = 1\right) \\ &= \prod_{y_{i1}+y_{i2}=1} \left(\frac{e^{(\mathbf{x}_{i2}-\mathbf{x}_{i1})^T\boldsymbol{\gamma}}}{1 + e^{(\mathbf{x}_{i2}-\mathbf{x}_{i1})^T\boldsymbol{\gamma}}}\right)^{y_i^*} \left(\frac{1}{1 + e^{(\mathbf{x}_{i2}-\mathbf{x}_{i1})^T\boldsymbol{\gamma}}}\right)^{1-y_i^*} \end{aligned}$$

where

$$y_i^* = \begin{cases} 1, & (y_{i1}, y_{i2}) = (0, 1) \\ 0, & (y_{i1}, y_{i2}) = (1, 0) \end{cases}$$